

# A colimit of classifying spaces

Graham Ellis

Mathematics Department, National University of Ireland, Galway

Roman Mikhailov

Steklov Mathematical Institute, Moscow

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## Abstract

We recall a group-theoretic description of the first non-vanishing homotopy group of a certain  $(n+1)$ -ad of spaces and show how it yields several formulae for homotopy and homology groups of specific spaces. In particular we obtain an alternative proof of J. Wu's group-theoretic description of the homotopy groups of a 2-sphere.

## 1 Introduction

This article is motivated by recent interest in describing homotopy and homology groups as intersections of subgroups (see for example [3, 5, 6, 10, 12, 17, 18, 23]). Some years ago the first author discovered a Hopf type formula for the higher integral homology of a group [15] using F. Keune's theory of relative nonabelian derived functors [19]. He subsequently observed that the formula could also be obtained as an easy consequence of a higher homotopy van Kampen theorem of R. Brown and J.-L. Loday [8] and published this topological proof with R. Brown in [7]. In this paper we take a closer look at the techniques underlying the topological proof with a view to pursuing ideas on intersections in free groups suggested by the second author and H.-J. Baues in [5].

Let  $G$  be a group with normal subgroups  $N_1, \dots, N_n$ . We are interested in the  $n$ th and  $(n+1)$ st homotopy groups of the topological space  $X$  arising as the homotopy colimit of classifying spaces  $B(G/\prod_{i \in I} N_i)$  where  $I$  ranges over all subsets  $I \subsetneq \{1, \dots, n\}$ . We need a connectivity condition and define an  $m$ -tuple of normal subgroups  $(N_1, \dots, N_m)$  to be *connected* if either  $m \leq 2$  or  $m \geq 3$  and for all subsets  $I, J \subset \{1, \dots, m\}$  with  $|I| \geq 2, |J| \geq 1$  the following equality holds:

$$\left( \bigcap_{i \in I} N_i \right) \left( \prod_{j \in J} N_j \right) = \bigcap_{i \in I} \left( N_i \left( \prod_{j \in J} N_j \right) \right). \quad (1)$$

Part of the statement of our main result involves a group  $T(N_1, \dots, N_n)$  which is recalled from [16] in Theorem 12 below. Our main result is:

**Theorem 1** *If the  $(n-1)$ -tuple  $(N_1, \dots, \hat{N}_i, \dots, N_n)$  is connected for each  $1 \leq i \leq n$  then the above colimit  $X$  has*

$$\pi_n(X) \cong \frac{N_1 \cap \dots \cap N_n}{\prod_{I \cup J = \{1, \dots, n\}, I \cap J = \emptyset} [\cap_{i \in I} N_i, \cap_{j \in J} N_j]}, \quad (2)$$

$$\pi_{n+1}(X) \cong \ker(\partial: T(N_1, \dots, N_n) \rightarrow G). \quad (3)$$

For  $n = 2$  isomorphism (2) is given in [6] and reads

$$\pi_2(X) \cong \frac{N_1 \cap N_2}{[N_1, N_2]}. \quad (4)$$

For  $n = 2$  isomorphism (3) is given in [8]. For  $n = 3$  isomorphism (2) is seemingly new and reads

$$\pi_3(X) = \frac{N_1 \cap N_2 \cap N_3}{[N_1, N_2 \cap N_3][N_2, N_1 \cap N_3][N_3, N_1 \cap N_2]}.$$

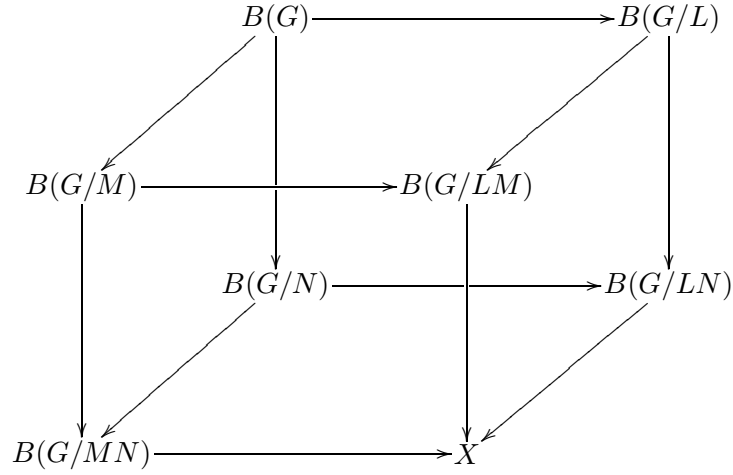
We shall explain how isomorphism (2) implies: the group-theoretic description of  $\pi_n(S^2)$  given by J. Wu [23]; homomorphisms of second and third homotopy groups given in [18] and [5] respectively; Hopf type formulae for the higher homology of a group given in [15, 7]; seemingly new results on the existence of torsion in certain groups arising from "almost aspherical" presentations.

Our approach is heavily influenced by work of R. Brown and J.-L. Loday. In [20] Loday introduced a functor  $\Pi: (\mathbf{n}\text{-cubes of spaces}) \rightarrow (\mathbf{cat}^n\text{-groups})$  from topology to algebra. Brown and Loday [8] proved that this functor preserves certain connectivity conditions and certain colimits. An explicit description of colimits in the algebraic category can be found in [6] for  $n = 1$ , in [20, 8] for  $n = 2$ , and in [16] for the general case  $n \geq 1$ . We shall explain how Theorem 1 is a fairly immediate consequence of this body of work. Indeed, Theorem 1 is really just an observation, but as this observation seems not to have been made before, and as it has a number of useful consequences, we give it the status of a theorem.

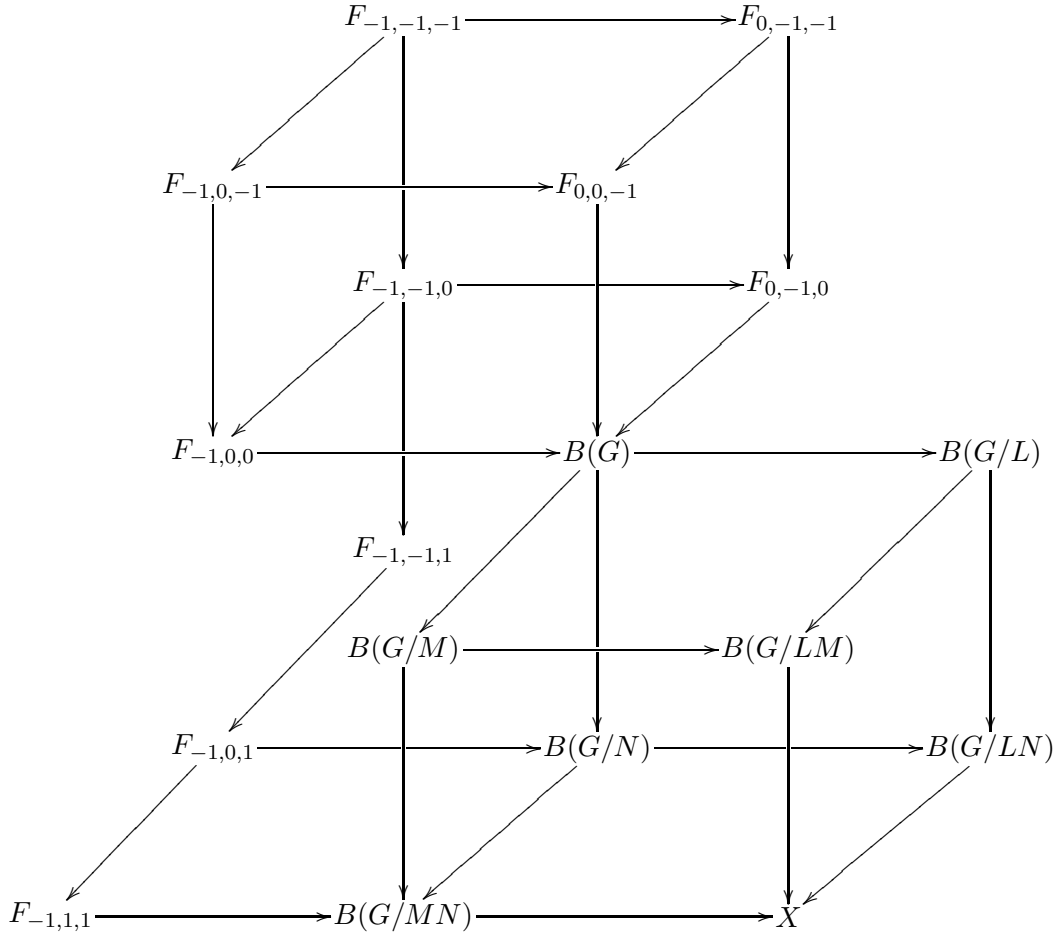
We begin with the case of three normal subgroups.

## 2 The case $n = 3$

Suppose that  $L, M, N$  are normal subgroups of a group  $G$ . This data gives rise to a commutative cube of spaces



in which  $B(G)$  denotes the classifying space of  $G$  and  $X$  is the homotopy pushout of the diagram of classifying spaces. Working up to homotopy type, we can extend this cube of spaces to a diagram of 27 spaces



in which each row and each column is a fibration sequence. For precise details of the construction see [20, 8].

The homotopy exact sequence of the fibration  $F_{-1,1,1} \rightarrow B(G/MN) \rightarrow X$  gives isomorphisms

$$\pi_n(X) \cong \pi_{n-1}(F_{-1,1,1}) \quad (n \geq 3). \quad (5)$$

The exact sequence of the fibration  $F_{-1,-1,1} \rightarrow F_{-1,0,1} \rightarrow F_{-1,1,1}$  gives isomorphisms

$$\pi_2(F_{-1,1,1}) \cong \ker(\pi_1(F_{-1,-1,1} \rightarrow L/(L \cap N)), \quad (6)$$

$$\pi_n(F_{-1,1,1}) \cong \pi_{n-1}(F_{-1,-1,1}) \quad (n \geq 4). \quad (7)$$

Note that

$$\pi_1(F_{-1,0,0}) \cong L \cap M. \quad (8)$$

**Notation.** For  $x, y \in G$  we define  $[x, y] = xyx^{-1}y^{-1}$  and  ${}^x y = xyx^{-1}$ . We write  $A \sqcup B = \langle n \rangle$  to mean that  $A, B$  are nonempty subsets of  $\{1, \dots, n\}$  with union equal to  $\{1, \dots, n\}$  and with trivial intersection. We write  $A \sqcup B \sqcup C = \langle n \rangle$  to mean that  $A, B, C$  are nonempty sets with union equal to  $\{1, \dots, n\}$  and with trivial pairwise intersections. We set  $N_1 = L, N_2 = M, N_3 = N$  and  $N_A = \cap_{i \in A} N_i$ .

Brown and Loday's higher van Kampen theorem [8] asserts that a functor  $\Pi$ : (3-cubes of spaces)  $\rightarrow$  (cat<sup>3</sup>-groups) preserves certain connectivity and colimits. In particular it asserts that all 27 spaces in the diagram are path-connected. The following consequence of the colimit property was proved in [13] and published in the form of a more general result in [16].

**Theorem 2** [13, 16] *The group  $\pi_1(F_{-1,-1,-1})$  is isomorphic to the group  $T(L, M, N)$  generated by symbols*

$$a \otimes_{A,B} b$$

for all  $A \sqcup B = \langle 3 \rangle, a \in N_A, b \in N_B$ , subject to the relations

$$a \otimes_{A,B} b = (b \otimes_{B,A} a)^{-1},$$

$$aa' \otimes_{A,B} b = ({}^a a' \otimes_{A,B} {}^a b)(a \otimes_{A,B} b),$$

$$({}^u[u^{-1}, v] \otimes_{U \cup V, W} {}^u w)({}^w[w^{-1}, u] \otimes_{W \cup U, V} {}^w v)({}^v[v^{-1}, w] \otimes_{V \cup W, U} {}^v u),$$

$$(a \otimes_{A,B} b)(a' \otimes_{A',B'} b')(a \otimes_{A,B} b)^{-1} = [a, b] a' \otimes_{A',B'} [a, b] b'$$

for  $A \sqcup B = A' \sqcup B' = \langle 3 \rangle, a \in N_A, a' \in N_{A'}, b \in N_B, b' \in N_{B'}, U \sqcup V \sqcup W = \langle 3 \rangle, u \in N_U, v \in N_V, w \in N_W$ . The homomorphism  $T(L, M, N) \rightarrow \pi_1(F_{-1,-1,0}) = L \cap M$  maps  $x \otimes y$  to the commutator  $[x, y]$ ; this homomorphism has the structure of a crossed module  $\partial: T(L, M, N) \rightarrow G, x \otimes y \mapsto [x, y]$  with action of  $g \in G$  given by  ${}^g(x \otimes y) = ({}^g x \otimes {}^g y)$ .

The homotopy exact sequence of the fibration  $F_{-1,-1,-1} \rightarrow F_{-1,-1,0} \rightarrow F_{-1,-1,1}$  together with isomorphisms (5)-(8) and Theorem 2 yield the following version of our main result for  $n = 3$ .

**Theorem 3** *There are isomorphisms*

$$\pi_3(X) \cong \frac{L \cap M \cap N}{[L, M \cap N][M, L \cap N][N, L \cap M]},$$

$$\pi_4(X) \cong \ker(\partial: T(L, M, N) \rightarrow G).$$

For completeness we also include isomorphisms for the first two homotopy groups of  $X$ ; these can both be deduced from Theorem 2 using the path-connectivity of the 27 spaces and the exact sequence of a fibration.

$$\pi_1(X) \cong \frac{G}{LMN}, \quad (9)$$

$$\pi_2(X) \cong \frac{LM \cap MN}{M(L \cap N)}. \quad (10)$$

Note that the left-hand side of (10) is symmetric in  $L, M, N$ . Hence the right-hand side is also symmetric in these three arbitrary normal subgroups of  $G$ .

The first isomorphism of the following corollary of Theorem 3 is a result of Wu [23].

**Corollary 4** *Let  $G = H * H$  be the free product of two copies of a group  $H$ . Let  $\lambda, \mu, \nu: H * H \rightarrow H$  be the first projection, second projection and multiplication homomorphisms respectively. Set  $L = \ker \lambda$ ,  $M = \ker \mu$ ,  $N = \ker \nu$ . Then the suspension  $SK(H, 1)$  of a classifying space for  $H$  has*

$$\pi_3(SK(H, 1)) = \frac{L \cap M \cap N}{[L, M \cap N][M, L \cap N][N, L \cap M]},$$

$$\pi_4(SK(H, 1)) \cong \ker(\partial: T(L, M, N) \rightarrow G).$$

**Proof.** Let  $B$  be a classifying space for  $H$ . Then the wedge  $B \vee B$  is a classifying space of  $G = H * H$ . Let  $CB$  denote the cone on  $B$ . Then  $CB \vee B$  is a classifying space for  $G/L$ ,  $B \vee CB$  is a classifying space for  $G/M$ , and  $B \times [0, 1]$  is a classifying space for  $G/M$ . Moreover, the pushout (or union) of these classifying spaces is a suspension  $X = SK(H, 1)$ . The corollary follows from Theorem 3.  $\square$

The first homomorphism of the following corollary was proved by Baues and Mikhailov [5].

**Corollary 5** *Let  $K_1, K_2, K_3$  be CW-subspaces of a connected CW-space  $K$  for which  $K = K_1 \cup K_2 \cup K_3$  and  $K^1 = K_1 \cap K_2 \cap K_3$  is the 1-skeleton of  $K$ . Let  $F = \pi_1 K^1$  and  $R_i = \ker(\pi_1 K^1 \rightarrow \pi_1 K_i)$  ( $i = 1, 2, 3$ ). Then there are homomorphisms*

$$\alpha_3: \pi_3(K) \longrightarrow \frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]},$$

$$\alpha_4: \pi_4(K) \longrightarrow \ker(\partial: T(R_1, R_2, R_3) \rightarrow F).$$

*In some cases both  $\alpha$  and  $\beta$  are isomorphisms.*

**Proof.** Each homomorphism  $\pi_1 K^1 \rightarrow \pi_1 K_i$  is surjective since the 1-skeleton of  $K_i$  equals the 1-skeleton  $K^1$  of  $K$ . Hence  $\pi_1 K_i \cong F/R_i$ . We can identify  $K^1$  with the classifying space  $B(F)$ . There are maps  $\iota_i: K_i \rightarrow B(F/R_i)$  inducing isomorphisms of fundamental groups. The maps  $\iota_i$  induce a map  $\iota: K = K_1 \cup K_2 \cup K_3 \rightarrow X$  where  $X$  is the colimit of Theorem 2 with  $G = F, L = R_1, M = R_2, N = R_3$ . The required homomorphisms  $\alpha_k$  are obtained by composing the isomorphisms of Theorem 2 with the induced homomorphisms  $\pi_k(\iota): \pi_k(K) \rightarrow \pi_k(X)$  ( $k = 3, 4$ ).

Consider the 2-sphere  $K = SK(\mathbb{Z}, 1)$  expressed as a union of  $K_1 = CS^1 \vee S^1$ ,  $K_2 = S^1 \vee CS^1$ ,  $K_3 = S * 1 \times [0, 1]$  as in the proof of Corollary 4. In this case the maps  $\alpha_3, \alpha_4$  are isomorphisms.  $\square$

One can obtain group-theoretic results using Theorem 3. Let  $\mathcal{X}$  be an alphabet and  $F = F(X)$  be the free group generated by  $\mathcal{X}$ . Let  $r_1, \dots, r_n$  be words in  $F$ . The group presentation  $\mathcal{P}(X, r_1, \dots, r_n) = \langle X \mid r_1, \dots, r_n \rangle$  is said to be *aspherical* if the associated 2-complex has trivial second homotopy group. We call the presentation *almost aspherical* if for every proper subset  $\mathcal{S}$  of the set  $\{r_1, \dots, r_n\}$  the presentation  $\langle X \mid \mathcal{S} \rangle$  is aspherical. For example, the 4-string braid group has the almost aspherical presentation  $\langle x, y, z \mid xyx = yxy, yzy = zyz, xz = zx \rangle$ .

**Corollary 6** *Let  $\mathcal{P}(X, r_1, r_2, r_3)$  be an almost aspherical presentation. Let  $R_i$  be the normal closure in  $F = F(X)$  of the relator  $\mathcal{R}_i$ . Then the quotient group*

$$\frac{F}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]}$$

*is torsion free.*

**Proof.** The 2-complexes associated to the presentations  $\langle X \mid r_i \rangle$  and  $\langle X \mid r_i, r_j \rangle$  are classifying spaces for  $F/R_i$  and  $F/R_i R_j$  ( $1 \leq i \neq j \leq 3$ ). So on taking  $G = F, L = R_1, M = R_2, N = R_3$  in Theorem 3 we can identify the colimit space  $X$  with the 2-complex associated to the three-relator presentation. Theorem 3 provides an exact sequence

$$0 \rightarrow \pi_3(X) \rightarrow \frac{F}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]} \rightarrow \frac{F}{R_1 \cap R_2 \cap R_3} \rightarrow 1.$$

The universal cover of  $X$  is homotopy equivalent to a wedge of 2-spheres and thus  $\pi_3(X)$  is known to be torsion free. The group  $F/R_1 \cap R_2 \cap R_3$  maps into the group  $(F/R_1) \times (F/R_2) \times (F/R_3)$  and each  $F/R_i$  is torsion free since it has an aspherical presentation. The corollary follows.  $\square$

It is easily shown that  $T(G, G, G)$  is isomorphic to the nonabelian symmetric square  $G \tilde{\otimes} G$  introduced by Dennis [11] and then Brown and Loday [8]. In light of this isomorphism, the following corollary can be found in [8].

**Corollary 7** *For the double suspension  $S^2 K(G, 1)$  of a classifying space  $K(G, 1)$  there are isomorphisms*

$$\begin{aligned} \pi_3(S^2 K(G, 1)) &\cong \frac{G}{[G, G]}, \\ \pi_4(S^2 K(G, 1)) &\cong \ker(\partial: T(G, G, G) \rightarrow G). \end{aligned}$$

**Proof.** It suffices to observe that, in the case  $L = M = N = G$ , the colimit  $X$  is of the homotopy type of  $S^2K(G, 1)$ .  $\square$

The above results on homotopy groups rely heavily on the homotopy exact sequence of a fibration. For calculations in group cohomology one defines  $B(G, N)$  to be the homotopy cofibre of the map  $B(G) \rightarrow B(G/N)$  and define  $H_k(G) = H_k(B(G), \mathbb{Z})$ ,  $H_k(G, N) = H_{k+1}(B(G, N), \mathbb{Z})$ . This yields a long exact sequence

$$\cdots \rightarrow H_{k+1}(G/N) \rightarrow H_{k+1}(G, N) \rightarrow H_k(G) \rightarrow H_k(G/N) \rightarrow \cdots .$$

One defines  $B(G, M, N)$  to be the homotopy cofibre of the canonical map  $B(G, N) \rightarrow B(G/M, NM/M)$  and  $H_k(G, M, N) = H_{k+2}(B(G, M, N), \mathbb{Z})$ . There is thus a long exact sequence

$$\cdots \rightarrow H_{k+1}(G/M, MN/M) \rightarrow H_{k+1}(G, M, N) \rightarrow H_k(G, N) \rightarrow H_k(G/M, MN/N) \rightarrow \cdots .$$

The isomorphism in the following corollary was originally proved by a purely algebraic argument using the theory of nonabelian left derived functors [15]; however, the first proof to appear in print was a short topological one [7]. We recall the topological proof here.

**Corollary 8** *Let  $R$  and  $S$  be normal subgroups of a free group  $F$  and suppose that both quotients  $F/R$  and  $F/S$  are free. Set  $G = F/RS$ . Then*

$$H_3(G) = \frac{R \cap S \cap [F, F]}{[R, S][R \cap S, F]}.$$

**Proof.** It is readily seen that, for any normal subgroups  $M, N$  in  $G$ , the cofibre  $B(G, M, N)$  is homotopy equivalent to the above colimit  $X$  in the case  $L = G$ . Theorem 3 shows that

$$\pi_3(B(G, M, N)) \cong \frac{M \cap N}{[G, M \cap N][M, N]}.$$

Since  $\pi_k(B(G, M, N)) = 0$  for  $k = 1, 2$ , the Hurewicz isomorphism  $\pi_3(B(G, M, N)) \cong H_3(B(G, M, N), \mathbb{Z})$  gives

$$H_1(G, M, N) \cong \frac{M \cap N}{[G, M \cap N][M, N]}.$$

Using  $H_k(F) = H_k(F/R) = H_k(F/S) = 0$  for  $k \geq 2$ , the exact homology sequences of a cofibration yield an isomorphism

$$H_3(G) \cong \ker(H_1(F, R, S) \rightarrow H_1(F)).$$

This yields the corollary.  $\square$

The above formula for  $H_1(G, M, N)$  is of independent interest. For a group-theoretic description of  $H_2(G, M, N)$  we define the group  $E(L, M, N)$  to be the quotient of  $T(L, M, N)$  obtained by imposing the extra relations

$$x \otimes x = 1$$

for all  $x \in L \cap M \cap N$ . There is an induced crossed module  $\partial: E(L, M, N) \rightarrow G$ . The following result was proved algebraically in [15]. To keep things topological we outline a topological proof of the second isomorphism.

**Corollary 9** *For a group  $G$  with normal subgroups  $M, N$  there are isomorphisms*

$$H_1(G, M, N) \cong \frac{M \cap N}{[G, M \cap N][M, N]},$$

$$H_2(G, M, N) \cong \ker(E(G, M, N) \rightarrow G).$$

**Proof.** Let  $L = G$ . We have already derived the first isomorphism from Theorem 3. For the second we consider J.H.C. Whitehead's exact sequence

$$H_5(X) \rightarrow \Gamma_4(X) \xrightarrow{\eta} \pi_4(X) \rightarrow H_4(X) \rightarrow 0.$$

A description of  $\Gamma_4(X)$  is given in [4]; in our case we find

$$\Gamma_4(X) = ((M \cap N)/[G, M \cap N][M, N]) \otimes \mathbb{Z}_2.$$

To complete the proof one has to check that  $\eta$  is induced by the homomorphism  $M \cap N \rightarrow T(G, M, N), x \mapsto x \otimes x$ .  $\square$

### 3 The case $n \geq 2$

We need to consider the category  $\{0 < 1\}$  with two objects and one non-identity arrow  $0 \rightarrow 1$ . Following [20] we define an  $n$ -cube of spaces to be a functor  $S: \{0 < 1\}^n \rightarrow (\text{Spaces})$ . We also need to consider the category  $\{-1 < 0 < 1\}$  with three objects and two non-identity arrows  $-1 \rightarrow 0$  and  $0 \rightarrow 1$ . An object in this category is an  $n$ -tuple  $\Delta \in \{0, 1\}^n$ ; we define a row to consist of a sequence of arrows  $\Delta' \rightarrow \Delta \rightarrow \Delta''$  where the  $n$ -tuples  $\Delta', \Delta, \Delta''$  are identical except in say the  $i$ th coordinate. Following [20] we define an  $n$ -cube of fibrations to be a functor  $F: \{-1 < 0 < 1\}^n \rightarrow (\text{Spaces})$  that maps each row to a fibration sequence. It is explained in [20, 8] how an  $n$ -cube of spaces  $S$  gives rise to an  $n$ -cube of fibrations  $F = \overline{S}$  where, up to homotopy type of maps, the restriction of  $\overline{S}$  to  $\{0 < 1\}^n$  is  $S$ .

**Definition 10** *Following [8] we say that an  $n$ -cube of spaces  $S$  is connected if each space in the  $n$ -cube of fibrations  $\overline{S}$  is path-connected.*

Given an  $n$ -tuple  $\Delta \in \{0, 1\}^n$  we shall write  $i \in \Delta$  to mean that the  $i$ th coordinate of  $\Delta$  equals 1. To  $n$  normal subgroups  $N_1, \dots, N_n$  of a group  $G$  we associate the  $n$ -cube of spaces  $B = B(G, N_1, \dots, N_n): \{0 < 1\}^n \rightarrow (\text{Spaces})$  in which

$$B_\Delta = B\left(\frac{G}{\prod_{i \in \Delta} N_i}\right)$$

is the classifying space of the quotient group  $G/\prod_{i \in \Delta} N_i$ , and the maps are induced by the canonical inclusions.

The verification of the following lemma, which uses the homotopy exact sequence of a fibration, is left to the reader.



**Lemma 11** *Let  $B = B(G, N_1, \dots, N_n)$  be the  $n$ -cube of spaces associated to  $n$  normal subgroups of  $G$ .*

- (i) *The  $n$ -cube  $B$  is connected if  $n = 1, 2$ .*
- (ii) *For  $n \geq 3$  the  $n$ -cube  $B$  is connected if and only if for all subsets  $I, J \subset \{1, \dots, n\}$  with  $|I| \geq 2, |J| \geq 1$  the following equality holds:*

$$\left( \bigcap_{i \in I} N_i \right) \left( \prod_{j \in J} N_j \right) = \bigcap_{i \in I} \left( N_i \left( \prod_{j \in J} N_j \right) \right).$$

Let  $B = B(G, N_1, \dots, N_n)$ . Let  $\Delta_{\text{final}} = (1, 1, \dots, 1)$  denote the  $n$ -tuple with each coordinate equal to 1. Let  $\Delta_{\text{initial}} = (-1, -1, \dots, -1)$  denote the  $n$ -tuple with each coordinate equal to  $-1$ . We are interested in the space  $X$  arising as the colimit of classifying spaces  $\{B_\Delta : \Delta \in \{0 < 1\}^n, \Delta \neq \Delta_{\text{final}}\}$ . Define  $C = C(G, N_1, \dots, N_n) : \{0 < 1\}^n \rightarrow (\text{Spaces})$  to be the  $n$ -cube of spaces obtained from  $B$  by replacing the classifying space  $B_{\Delta_{\text{final}}}$  with the colimit  $X$ . Let  $F = \overline{C}$  be the  $n$ -cube of fibrations associated to  $C$ . The following theorem is a special case of Theorem 2.3 in [16] and is a consequence of Brown and Loday's higher van Kampen theorem.

**Theorem 12** [16] *Let  $N_1, \dots, N_n$  be normal subgroups of  $G$  such that the  $(n-1)$ -cube of spaces  $B(G, N_1, \dots, \hat{N}_i, \dots, N_n)$  is connected for each  $1 \leq i \leq n$ . Let  $F = \overline{C}$  be the  $n$ -cube of fibrations associated to  $C = C(G, N_1, \dots, N_n)$ . Every space in  $F$  is path-connected and  $\pi_1(F_{\Delta_{\text{initial}}})$  is isomorphic to the group  $T(N_1, \dots, N_n)$  generated by symbols*

$$a \otimes_{A,B} b$$

*for all  $A \sqcup B = \langle n \rangle, a \in N_A, b \in N_B$ , subject to the relations*

$$\begin{aligned} a \otimes_{A,B} b &= (b \otimes_{B,A} a)^{-1}, \\ aa' \otimes_{A,B} b &= ({}^a a' \otimes_{A,B} {}^a b)(a \otimes_{A,B} b), \\ ({}^u[u^{-1}, v] \otimes_{U \cup V, W} {}^u w)({}^w[w^{-1}, u] \otimes_{W \cup U, V} {}^w v)({}^v[v^{-1}, w] \otimes_{V \cup W, U} {}^v u), \\ (a \otimes_{A,B} b)(a' \otimes_{A', B'} b')(a \otimes_{A,B} b)^{-1} &= [a, b] a' \otimes_{A', B'} [a, b] b' \end{aligned}$$

*for  $A \sqcup B = A' \sqcup B' = \langle n \rangle, a \in N_A, a' \in N_{A'}, b \in N_B, b' \in N_{B'}, U \sqcup V \sqcup W = \langle n \rangle, u \in N_U, v \in N_V, w \in N_W$ . The homomorphism  $T(N_1, \dots, N_n) \rightarrow \pi_1(F_{-1, -1, \dots, -1, 0}) \leq G$  maps  $x \otimes y$  to the commutator  $[x, y]$ ; this homomorphism has the structure of a crossed module  $\partial : T(N_1, \dots, N_n) \rightarrow G, x \otimes y \mapsto [x, y]$  with action of  $g \in G$  given by  ${}^g(x \otimes y) = ({}^g x \otimes {}^g y)$ .*

**Proof of Theorem 1.** Our derivation of Theorem 3 from Theorem 2 extends routinely to yield our main Theorem 1 as a corollary of Theorem 12. One simply has to apply the homotopy exact sequence of a fibration  $n$  times. Lemma 11 provides the algebraic version of the connectivity condition.  $\square$

As already mentioned, Theorem 1 for the case  $n = 2$  is given in [8]. This case yields the following earlier result of Brown [6]

**Corollary 13** [6] *Given two normal subgroups  $M, N$  in a group  $G$  we have*

$$\pi_2(B(G/M) \cup_{B(G)} B(G/N)) = \frac{M \cap N}{[M, N]}.$$

Corollary 13 implies that  $R_i \cap R_j = [R_i, R_j]$  ( $i \neq j$ ) in Corollary 6. Corollary 13 also implies that  $L \cap M = [L, M]$ ,  $M \cap N = [M, N]$ ,  $L \cap N = [L, N]$  in Corollary 4. Furthermore, Corollary 13 can be used to recover the exact sequence of second homotopy groups proved by M. Gutierrez and J. Ratcliffe in [18].

With one exception, the corollaries in Section 3 and their proofs extend to analogous results involving  $n \geq 2$  normal subgroups. The exception is Corollary 8 where we do not know an explicit description of Whitehead's sequence in higher dimensions necessary for extending the topological proof.

## 4 Group-theoretical applications

Some unexpected results in group theory arise from well-known results in homotopy theory. For example Corollary 6 generalizes to the following.

**Corollary 14** *Let  $\mathcal{P}(X, r_1, \dots, r_n)$  be an almost aspherical presentation. Let  $R_i$  be the normal closure in  $F = F(X)$  of the relator  $R_i$ . Then the quotient group*

$$\frac{F}{\prod_{I \cup J = \{1, \dots, n\}, I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j]}$$

*can only have  $p$ -torsion if the  $n$ th homotopy group of a wedge of spheres can have  $p$ -torsion. So for example, by a result of Serre, the group can have no  $p$  torsion for primes  $p > 2n$ .*

**Proof** Consider a proper subset  $I \subsetneq \{1, \dots, n\}$ . Since the group presentation  $\mathcal{P}(X, r_i \ (i \in I))$  is aspherical the associated 2-complex  $K_I$  is a classifying space for the group  $F / \prod_{i \in I} R_i$ . Also, this space  $K_I$  is the homotopy colimit (i.e. union) of the diagram of spaces  $\{B(F / \prod_{i \in J} R_j) : J \subsetneq I\} = \{K_J : J \subsetneq I\}$ . Using induction on the size of  $I$  together with the connectivity assertion of Theorem 12 we obtain that the  $|I|$ -cube of spaces  $B(F, R_i \ (i \in I))$  is connected. In particular each  $(n-1)$ -cube  $B(F, R_1, \dots, \hat{R}_i, \dots, R_n)$  is connected and so we can apply Theorem 1 to the normal subgroups  $(F, R_1, \dots, R_n)$ ; we get a formula for the  $n$ -th homotopy group of a colimit space which, in this case, is the 2-complex  $K$  associated to the presentation  $\mathcal{P}(X, r_1, \dots, r_n)$ . Now  $\pi_2 K \cong \pi_2 \tilde{K}$  and  $\tilde{K}$  is a wedge of 2-spheres. The remainder of the proof is now just a copy of the proof of Corollary 6.  $\square$

For a second example we recall a description of the homotopy groups of the 2-sphere due to Wu [23]. Let  $F[S^1]$  be Milnor's  $F[K]$ -construction applied to the simplicial circle  $S^1$ . This is the free simplicial group with  $F[S^1]_n$  a free group of rank  $n \geq 1$  with generators  $x_0, \dots, x_{n-1}$ . Changing the basis of  $F[S^1]_n$  in the following way:  $y_i = x_i x_{i+1}^{-1}$ ,  $y_{n-1} = x_{n-1}$ , we get another basis  $\{y_0, \dots, y_{n-1}\}$  in which the

simplicial maps can be written more easily. A combinatorial group-theoretical argument then gives a description of the  $n$ -th homotopy group of the loop space  $\Omega\Sigma S^1$ , which is isomorphic to the homotopy group of  $\pi_{n+1}(S^2)$  (see [23] for precise details); one finds Wu's formula

$$\pi_{n+1}(S^2) \cong \frac{\langle y_{-1} \rangle^F \cap \langle y_0 \rangle^F \cap \cdots \cap \langle y_{n-1} \rangle^F}{[[y_{-1}, y_0, \dots, y_{n-1}]]}, \quad n \geq 1 \quad (11)$$

where  $F$  is a free group with generators  $y_0, \dots, y_{n-1}$ ,  $y_{-1} = (y_0 \dots y_{n-1})^{-1}$ , and  $[[y_{-1}, y_0, \dots, y_{n-1}]]$  is the normal closure in  $F$  of the set of left-ordered commutators

$$[z_1^{\varepsilon_1}, \dots, z_t^{\varepsilon_t}] \quad (12)$$

with the properties that  $\varepsilon_i = \pm 1$ ,  $z_i \in \{y_{-1}, \dots, y_{n-1}\}$  and all elements in  $\{y_{-1}, \dots, y_{n-1}\}$  appear at least once in the sequence of elements  $z_i$  in (12).

Consider  $n \geq 1$  and Milnor's construction  $F[S^n]$ . The lower terms of the simplicial group  $F[S^n]$  are the following:

$$\begin{aligned} F[S^n]_n &= F(\sigma), \\ F[S^n]_{n+1} &= F(s_0\sigma, \dots, s_n\sigma), \\ F[S^n]_{n+2} &= F(s_j s_i \sigma \mid i < j), \\ &\dots \end{aligned}$$

For  $n \geq 1$ , the generator of

$$\pi_{n+1}(F[S^n]) \simeq \pi_{n+1}(\Omega(S^{n+1})) \simeq \pi_{n+2}(S^{n+1})$$

can be chosen as the commutator  $[s_0\sigma, s_1\sigma]$  in  $F[S^n]_{n+1}$  (see [23]). Now these commutators considered as elements of  $F[S^n]_{n+1}$  define the elements from  $F[S^1]_k$ , which correspond the homotopy classes of composition maps

$$S^{k+1} \rightarrow S^k \rightarrow \dots \rightarrow S^3 \rightarrow S^2,$$

where every map is viewed as a suspension over the Hopf fibration. Let us consider these elements.

1. First, let  $F_2 = F(y_0, y_1)$ , then the element

$$[y_0, y_1] \notin [[y_{-1}, y_0, y_1]]$$

corresponds to the homotopy class of the Hopf fibration  $S^3 \rightarrow S^2$ .

2. Let  $F_3 = F(y_0, y_1, y_2)$ , then the element

$$[[y_0, y_1], [y_0, y_1 y_2]] \notin [[y_{-1}, y_0, y_1, y_2]]$$

corresponds to the homotopy class of the composition map  $S^4 \rightarrow S^3 \rightarrow S^2$ .

3. Let  $F_4 = F(y_0, y_1, y_2, y_3)$ , then the element

$$[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3]]] \notin [[y_{-1}, y_0, y_1, y_2, y_3]]$$

corresponds to the homotopy class of the composition map  $S^5 \rightarrow S^4 \rightarrow S^3 \rightarrow S^2$ .

4. Let  $F_5 = F(y_0, y_1, y_2, y_3, y_4)$ , then the element

$$\begin{aligned} &[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3]]], \\ &[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3 y_4]]]] \\ &\notin [[y_{-1}, y_0, y_1, y_2, y_3, y_4]] \end{aligned}$$

corresponds to the homotopy class of the composition map  $S^6 \rightarrow S^5 \rightarrow S^4 \rightarrow S^3 \rightarrow S^2$ .

5. Let  $F_6 = F(y_0, y_1, y_2, y_3, y_4, y_5)$ , then the element

$$\begin{aligned} &[[[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3]]], \\ &[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3 y_4]]]], \\ &[[[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3]]], \\ &[[[y_0, y_1], [y_0, y_1 y_2]], [[y_0, y_1], [y_0, y_1 y_2 y_3 y_4 y_5]]]]]] \\ &\in [[y_{-1}, y_0, y_1, y_2, y_3, y_4, y_5]] \end{aligned}$$

corresponds to the trivial homotopy class of the composition map

$$S^7 \rightarrow S^6 \rightarrow S^5 \rightarrow S^4 \rightarrow S^3 \rightarrow S^2. \quad (13)$$

The triviality of this map can be proved using standard methods in homotopy theory [22]. This is the simplest case of the Nilpotence Theorem due to Nishida [21], which states that every element in the ring of stable homotopy groups of spheres is nilpotent.

We now turn to a group-theoretic application of Theorem 1. Consider a free group  $F(X)$  with a basis  $X$  and let  $\mathcal{S} = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$  be a collection of sets  $\mathcal{R}_i$  of words, such that: 1) the presentation

$$\langle X \mid \mathcal{R}_1, \dots, \mathcal{R}_n \rangle \quad (14)$$

is not aspherical, 2) the presentation

$$\langle X \mid \mathcal{R}_{i_1}, \dots, \mathcal{R}_{i_k} \rangle \quad (15)$$

is aspherical for every proper subset  $\{\mathcal{R}_{i_1}, \dots, \mathcal{R}_{i_k}\}$  of  $\mathcal{S}$ . Such a presentation we shall call  $\mathcal{S}$ -almost aspherical. Observe that every almost aspherical presentation is  $\mathcal{S}$ -almost aspherical, where  $\mathcal{S}$  is the collection of all relators, i.e. the sets  $\mathcal{R}_i$  contain single relators for all  $i$ .

Denote by  $R_i$  the normal closure of  $\mathcal{R}_i$  for  $i = 1, \dots, n$ . Observe that the standard 2-complex, associated to an  $\mathcal{S}$ -almost aspherical presentation (14) is homotopically equivalent to the homotopy colimit  $X$  of the  $n$ -cube of standard 2-complexes associated with subpresentations of the type (15), i.e. to the colimit of the non-final spaces in the  $n$ -cube  $B(F, R_1, \dots, R_n)$ . The asphericity condition 2) implies that the above

$n$ -cube is  $n$ -connected and hence the  $n$ -th homotopy group  $\pi_n(X)$  can be computed by formula given in Theorem 1. However, the colimit  $X$  is homotopically equivalent to a (non-empty) wedge of 2-spheres and, therefore, formulas for the homotopy groups of wedges of 2-spheres follow.

As an example, consider an almost aspherical presentation of the trivial group:

$$\mathcal{P} = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n+1} \rangle$$

with  $n$  generators and  $n + 1$  relators ( $n \geq 1$ ). Let  $K_{\mathcal{P}}$  be the standard 2-complex associated with  $\mathcal{P}$ . The chain complex  $C_*K_{\mathcal{P}}$  has  $C_2K_{\mathcal{P}} = \mathbb{Z}^{n+1}$ ,  $C_1K_{\mathcal{P}} = \mathbb{Z}^n$  and natural epimorphism  $C_2K_{\mathcal{P}} \rightarrow C_1K_{\mathcal{P}}$ , which kernel is isomorphic to  $H_2(K_{\mathcal{P}}) = \pi_2(K_{\mathcal{P}}) = \mathbb{Z}$ , hence  $K_{\mathcal{P}}$  is homotopically equivalent to the 2-sphere  $S^2$ . Theorem 1 implies the following presentation of the  $(n + 1)$ -st homotopy group of  $S^2$ :

$$\pi_{n+1}(S^2) \simeq \frac{R_1 \cap \dots \cap R_{n+1}}{\prod_{I \cup J = \{1, \dots, n\}, I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j]}, \quad (16)$$

where  $R_i = \langle r_i \rangle^F$ ,  $F$  being the free group with generators  $x_1, \dots, x_n$ . Clearly, in this case, the subgroup of  $F / (\prod_{I \cup J = \{1, \dots, n\}, I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j])$  given in the right hand side of (16) is central. In particular, consider the presentation

$$\mathcal{P}_{WU} = \langle x_1, \dots, x_n \mid x_1, \dots, x_n, x_1 \cdots x_n \rangle.$$

It is easy to see that  $\mathcal{P}_{WU}$  is almost aspherical. A standard commutator calculus argument, given essentially in Corollary 3.5 of [23] shows that

$$[[x_1 \dots x_n, x_1, \dots, x_n]] = \prod_{I \cup J = \{1, \dots, n+1\}, I \cap J = \emptyset} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j], \quad (17)$$

and Wu's isomorphism (11) follows from (16). (One could also obtain (17) using Theorem 1.)

Now consider an arbitrary set of elements  $r_1, \dots, r_{n+1}$  in the free group  $F$  with basis  $x_1, \dots, x_n$  and with the following property: the groups defined by presentations with  $n$  generators and  $n$  relators

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_{j-1}, \hat{r}_j, r_{j+1}, \dots, r_{n+1} \rangle$$

are trivial for all  $j = 1, \dots, n + 1$  (i.e. we consider the presentation without symbol  $r_j$ ). Since every balanced presentation, i.e. a presentation with equal number of generators and relators, of the trivial group is aspherical, we have the following: *either  $\pi_{n+1}(S^2)$  can be presented as (16) for  $R_i = \langle r_i \rangle^F$ ,  $i = 0, \dots, n + 1$ , or there is a counter-example to Whitehead asphericity conjecture, i.e. there exists an aspherical 2-dimensional complex with non-aspherical subcomplex.*

Since every subcomplex of an aspherical complex, with a single 2-cell, is aspherical (see [1]), we obtain the following. Let  $F$  be a free group with generators  $x_1, x_2$  and let  $r_1, r_2, r_3$  be words in  $F$  such that the groups  $F/R_1R_2, F/R_1R_3, F/R_2R_3$  are trivial for  $R_i = \langle r_i \rangle^F$ ,  $i = 1, 2, 3$ . Then there is an isomorphism

$$\frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]} \simeq \mathbb{Z}.$$

As an example, consider

$$r_1 = x_1^2 x_2^{-3}, \quad r_2 = x_1^3 x_2^{-4}, \quad r_3 = x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1}.$$

The groups

$$\langle x_1, x_2 \mid x_1^n x_2^{-(n+1)}, \quad x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1} \rangle$$

are trivial for  $n \geq 2$ . The above presentation of trivial groups is due to Akbulut-Kirby [2]. It is easy to see that the presentation

$$\langle x_1, x_2 \mid x_1^2 x_2^{-3}, \quad x_1^3 x_2^{-4} \rangle$$

defines the trivial group. It would be interesting to find a natural generalization of this presentation to the case of  $n$  generators and  $n + 1$  relators and to consider group-theoretical meaning of the fact that the composition map (13) is homotopically trivial.

## 5 Other varieties

The Hopf type formula of Corollary 8 and its higher dimensional analogues were first proved using a modification of a technique of F. Keune for nonabelian derived functors. The technique is very general and can be applied in situations such as the homology of Lie algebras or varietal Baer invariants of groups. Indeed, it was used in [9] to derive a Hopf type formula for the Baer invariant corresponding to the variety of 2-nilpotent groups; this formula was implemented on a computer and applied to all groups of order at most 30.

The nonabelian derived functor approach and the generalised van Kampen theorem approach to Hopf type formulae both apply only to connected  $n$ -cubes (see Definition 10 and Lemma 11). Unfortunately this hypothesis was omitted from the statement of theorems in [15] and [7] and some subsequent publications. This seems to have caused confusion. However, in all calculations based on the theorems the connectivity condition was met. A detailed erratum [14] has been available on the first author's home page since 2002.

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